Lecture 19

Friday, March 6, 2020 6:32 AM

Lewy extension of CR fcns.

- Recall that if $M \subseteq \mathbb{C}^n$ is a real $\mathcal{C}^k$ hypersurface, i.e. $M = \{z : \rho(z) = 0\}$, $\rho \in \mathcal{C}^k(\mathbb{C}^n, \mathbb{R})$, $d\rho|_M \neq 0$ (e.g. $M = \partial\Omega$, $\Omega \subseteq \mathbb{C}^n$ w/ smooth bdry), then $u \in \mathcal{C}^k(M)$ is <u>CR</u> if $\forall\, \zeta \in M$, $X \in T^{1,0}_\zeta M$ $\left(\Leftrightarrow X \in \mathbb{C}^n \text{ s.t. } \sum_{i=1}^n \frac{\partial \rho}{\partial z_i}(\zeta) X^i = 0\right)$ we have $\sum_{i=1}^n \overline{X^i}\, u_{\bar z_i}(\zeta) = 0$. (Introduced when we discussed Bochner's Thm.)

- Equivalently $u \in \mathcal{C}^k$ in some open nbhd of M and $\bar\partial u \wedge \partial\rho = 0$ on M.

<u>Lewy's Extension Thm</u>. Let $M \subseteq \mathbb{C}^n$ be real $\mathcal{C}^k$ ($k \geq 4$) hypersurface w/ defining function ρ. Suppose $\zeta \in M$ and $\exists\, X \in T^{1,0}_\zeta M$ s.t. $\sum_{i,j} \rho_{z_i \bar z_j}(\zeta) X^i \overline{X^j} < 0$. Then, $\exists\, \zeta \in \omega' \subseteq \omega \subseteq \mathbb{C}^n$ s.t. $\forall\, u \in \mathcal{C}^k(M \cap \omega) \cap CR$ $\exists\, U \in \mathcal{O}(\omega'_+) \cap \mathcal{C}^{k-3}(\overline{\omega'_+})$ s.t. $U = u$ on $M \cap \omega'$. Here, $\omega'_+ = \{z \in \omega' : \rho(z) > 0\}$.

<u>Pf.</u> • After affine change of coordinates WLOG $\zeta = 0$, $\rho(z) = \operatorname{Im} z_n + \operatorname{Re} \sum_{i,j=1}^n A_{ij} z^i z^j + \sum_{i,j} B_{i\bar j} z^i \overline{z^j} + O(|z|^3)$; $A_{ij} = \rho_{z_i z_j}(0)$; $B_{i\bar j} = \rho_{z_i \bar z_j}(0)$. ($A$ symm., B Herm.)

• Another c.o.c. $z_j = \tilde z_j$, $j = 1, \dots, n-1$, $z_n = \tilde z_n - i \sum_{i,j} A_{ij} \tilde z^i \tilde z^j$ $\Rightarrow$ (drop ~ on $\tilde z$)

$$\rho(z) = \operatorname{Im} z_n + \sum_{i,j} B_{i\bar j} z^i \overline{z^j} + \underbrace{O(|z|^3)}_{\text{abbreviate } O(3)}.$$

$X \in T^{1,0}_0 M \Leftrightarrow X_n = 0$. Thus, assumption is $\exists\, X' \in \mathbb{C}^{n-1}$ s.t. $\sum_{i,j}^{n-1} B_{i\bar j} (X')^i \overline{(X')^j} < 0$

WLOG $X' = (1, 0, \dots, 0) \Rightarrow B_{1\bar 1} < 0$. Hence, $\rho(z_1, 0, \dots, 0) = B_{1\bar 1}|z_1|^2 + O(|z_1|^3)$

$\Rightarrow \exists\, \delta > 0$ and $c > 0$ s.t. $\rho(z_1, 0, \dots, 0) \leq -c|z_1|^2$ in $|z_1| \leq \delta$.

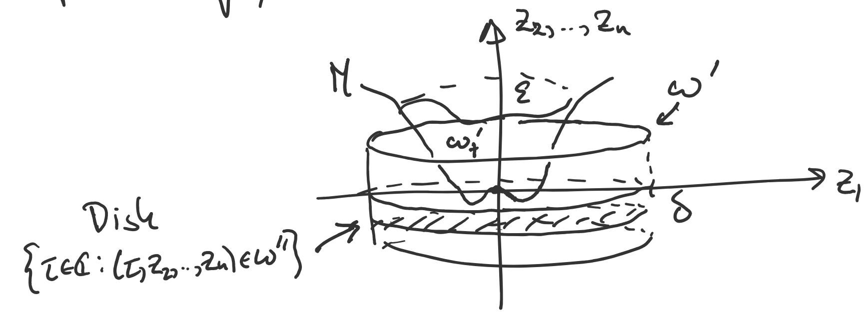
$\Rightarrow \exists\, \varepsilon > 0$ s.t. $\rho(z) < 0$ in $\frac{\delta}{2} \leq |z_1| \leq \delta$ and $|z_2| < \varepsilon, \dots, |z_n| < \varepsilon$.

WLOG; $\delta, \varepsilon > 0$ so small that ω'_+, ω'_- are connected.

Let $\omega' = \{|z_1| < \delta, |z_2| < \varepsilon, \dots, |z_n| < \varepsilon\}$. Then $\omega'_+ \subseteq \{|z_1| < \delta/2\}$. Moreover, for $y \in \mathbb{R}$

Let $\omega' = \{|z_1|<\delta, |z_2|<\varepsilon, \dots, |z_n|<\varepsilon\}$. Then $\omega_+ \subseteq \{|z_1| < \delta/2\}$. Moreover, ...

$\rho(z_1, 0, iy) \le y - c|z_1|^2 + B_{nn}y^2 + O(3) \Rightarrow \rho(z_1,0,iy) < \frac{y}{2}$ in $|z_1| \le \delta$ for $-\varepsilon < y < 0$ (by shrinking ε if necessary) $\Rightarrow$ $\omega'' = \{|z_1|<\delta, |z_2|<\varepsilon, \dots, |z_{n-1}|<\varepsilon, |z_n - iy_0| < \varepsilon'\}$ $= \mathbb{D}_\delta \times \tilde{\omega}'' \subseteq \omega'_-$.



Now, we mimic pf of Bochner + pf of $\bar\partial$-thm w/cpt supp.

$\bar\partial u = h_0 \partial\rho + h_1\rho$

We may assume that $u = U_0|_M$, $U_0 \in \mathcal{C}^k(\omega)$. Since $\bar\partial u \wedge \bar\partial\rho = 0$ on M, we can assume $\bar\partial U_0 = O(\rho^2)$ (almost hol. extension). We can then define $f \in \mathcal{C}^1_{(0,1)}(\omega')$ s.t. $f = -\bar\partial U_0$ in ω'_+ and $f = 0$ in ω'_-. Thus, $\bar\partial f = 0$ in ω^+. So $f = \sum_i f_i \, d\bar z_i$, $f_{i,\bar z_j} = f_{j, \bar z_i}$. We set

$$v(z) = \frac{1}{2\pi i} \int_{|\tau| \le \delta} \frac{f_1(\tau, z_2, \dots, z_n)}{\tau - z_1} \, d\tau \wedge d\bar\tau .$$

As in pf of $\bar\partial$-Thm, since $f = 0$ in $\frac{\delta}{2} \le |z_1| \le \delta$, we conclude $\bar\partial v = f$ in ω'. Thus, $v \in \mathcal{O}(\omega'_-)$ and since $f(\tau, z_2, \dots, z_n) = 0$ for $|\tau| \le \delta$, $(z_2, \dots, z_n) \in \tilde{\omega}''$, $v = 0$ in $\omega'' \subseteq \omega'_-$ connected. $\Rightarrow v = 0$ on M.

Set $U = U_0 + v$. Then $U \in \mathcal{C}^1(\omega')$, $\bar\partial U = 0$ in ω'_+ $(U \in \mathcal{O}(\omega'_+))$ and $U|_M = U_0|_M = u$.

∎